

Quantum interest in two dimensions

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Abstract

The quantum interest conjecture of Ford and Roman asserts that any negative-energy pulse must necessarily be followed by an over-compensating positive-energy one within a certain maximum time delay. Furthermore, the minimum amount of over-compensation increases with the separation between the pulses. In this paper, we first study the case of a negative-energy square pulse followed by a positive-energy one for a minimally coupled, massless scalar field in two-dimensional Minkowski space. We obtain explicit expressions for the maximum time delay and the amount of over-compensation needed, using a previously developed eigenvalue approach. These results are then used to give a proof of the quantum interest conjecture for massless scalar fields in two dimensions, valid for general energy distributions.

1. Introduction

Most forms of classical matter obey the weak energy condition (WEC) [1]. This means that the local energy density $\rho(t)$, as measured by an observer with proper time t , is always non-negative. The WEC has a number of important implications in general relativity; for example, it can be shown that matter obeying the WEC will necessarily form a singularity after a certain critical stage in gravitational collapse [2, 1]. On the other hand, matter violating the WEC would lead to a host of exotic predictions, such as the existence of traversable wormholes [3] and time machines [4], naked singularities [5, 6], and ‘faster-than-light’ travel [7, 8].

As it turns out, the WEC is violated in quantum field theory. A classic result of Epstein, et al. [9] states that the (renormalised) energy density of a quantum field at a given space-time point can be arbitrarily negative. However, there are constraints coming in when the energy density is averaged over time. One such constraint is the averaged weak energy condition (AWEC), which states that the integral of the energy density over the observer’s geodesic is non-negative [10]:

$$\int_{-\infty}^{\infty} \rho(t) dt \geq 0. \quad (1.1)$$

The AWEC is known to hold for a number of quantum fields in Minkowski space [11].

More recently, a new class of constraints, known as *quantum inequalities* (QI’s), was derived by Ford and Roman [12, 13, 14]. QI’s provide lower bounds on the weighted average of the energy density seen by a geodesic observer:

$$\int_{-\infty}^{\infty} \rho(t) f(t) dt \geq \rho_{\min}, \quad (1.2)$$

where the weighting is supplied by a ‘sampling function’ $f(t)$, i.e., a peaked function of time with unit integral and a certain characteristic width t_0 . The lower bound ρ_{\min} is a negative quantity that depends on a number of parameters, including the type of quantum field, the space-time it is defined on, and the sampling function used. For example, the QI for minimally coupled, massless scalar fields in $2n$ -dimensional Minkowski space takes the general form [15, 16]

$$\int_{-\infty}^{\infty} \rho(t) f(t) dt \geq -\frac{1}{c_n} \int_{-\infty}^{\infty} \left| \left(\frac{d}{dt} \right)^n f^{1/2}(t) \right|^2 dt, \quad (1.3)$$

where c_n is a positive constant given by

$$c_n \equiv \begin{cases} 6\pi & n = 1; \\ n\pi^{n-1/2} 2^{2n} \Gamma(n - \frac{1}{2}) & n \geq 2. \end{cases} \quad (1.4)$$

QI's have also been derived for a number of other cases [17, 18, 19, 20, 21, 22, 23, 24].

Now, for example, if the Lorentzian sampling function $f(t) = t_0/[\pi(t^2 + t_0^2)]$ is used in (1.3), it can be shown that $\rho_{\min} \propto -t_0^{-2n}$. Thus, in the infinite sampling time limit, $\rho_{\min} \rightarrow 0^-$, and so the QI reduces to the AWEC. This implies, in particular, that any negative-energy distribution must necessarily be followed (or preceded) in time by at least the same amount of positive energy. Ford and Roman [6, 25] have likened this to bank loans: the negative energies are loans, and the positive parts repayments. The AWEC then becomes the statement that all loans must be repaid in full over time.

Ford and Roman have investigated this phenomenon for the case of δ -function pulses in two- and four-dimensional massless scalar field theory [25], for a specific choice of sampling function. They showed that a negative δ -function pulse must necessarily be followed by a positive one within a certain maximum time delay. Furthermore, they found that the positive δ -function pulse must *over*-compensate the negative one, by an amount which monotonically increases with the separation between the pulses. Ford and Roman have conjectured that this is, in fact, a general phenomenon of quantum fields, and have termed it the *quantum interest conjecture*: the negative energy loan must be repaid with a positive interest ‘rate’; moreover there is a maximum term for the loan.

It is of obvious interest to check the quantum interest conjecture for more general energy distributions and sampling functions. Pretorius [26] has managed to extend the results in the former direction. However, as with Ford and Roman’s analysis, a specific one-parameter family of sampling functions was assumed, and the appropriate QI optimised over it. This is, unfortunately, not guaranteed to give the best possible bounds for the interest ‘rate’ and maximal loan term.

Fewster and one of the present authors [27] have recently developed a new approach to quantum interest for massless scalar fields in even dimensions, which involves turning it into an eigenvalue problem familiar from quantum mechanics. This is actually quite easy to see if we gloss over some technical issues (which are rigorously addressed in [27]). If we write $\psi(t) = f^{1/2}(t)$ and integrate by parts n times, (1.3) can be recast in the form

$$\langle \psi | H^{(n)} \psi \rangle \geq 0, \quad (1.5)$$

where $\langle \cdot | \cdot \rangle$ is the usual L^2 -inner product, and $H^{(n)}$ is the differential operator

$$H^{(n)} \equiv (-1)^n \left(\frac{d}{dt} \right)^{2n} + c_n \rho(t). \quad (1.6)$$

The condition (1.5) is equivalent to the requirement that $H^{(n)}$ does *not* have any negative eigenvalues.

Hence, in this approach, the energy density $\rho(t)$ becomes (up to a constant) the potential of a generalised Schrödinger operator $H^{(n)}$, and the non-existence of negative eigenvalues to $H^{(n)}$ is equivalent to the QI being satisfied for $\rho(t)$. Note that the foregoing condition does not make any reference to the ‘wavefunction’ $\psi(t)$. Any $H^{(n)}$ obeying this condition is therefore guaranteed to give a $\rho(t)$ satisfying (1.3) for *any* sampling function $f(t)$, and we shall refer to it as a *QI-compatible* energy density, following [27].

In two-dimensional Minkowski space, it happens that the operator $H^{(1)} = -d^2/dt^2 + 6\pi\rho(t)$ has precisely the form of the Hamiltonian in quantum mechanics. This would allow us to turn the problem of quantum interest in two dimensions into a one-dimensional quantum-mechanical problem, where time is now the spatial coordinate. The negative parts of $\rho(t)$ become potential wells, and the positive parts potential barriers. Given a form of the energy distribution, one can in principle find out whether bound states (with negative eigenvalues) exist for this potential, and therefore whether the energy distribution is QI-compatible or not.

The example of double δ -function pulses was treated in [27] using this approach, albeit rather briefly, and bounds on the maximal loan term and interest ‘rate’ were obtained. These bounds are tighter than those obtained in [25], since they are automatically optimised over all possible sampling functions, and are limited only by how optimal the original QI used is. As it fortuitously turns out, the QI (1.3) *is* optimal in two dimensions [15].

In the first part of this paper, we shall consider another example, namely that of double square pulses, and show in detail how the eigenvalue approach can be used to extract optimal bounds on the maximal loan term and interest ‘rate’. This case turns out to exhibit a much richer behaviour than the δ -function pulse case, yet it is simple enough to be solved analytically.

Of course, it would be desirable to eventually move beyond specific examples, and see how the eigenvalue approach can be used to prove the quantum interest conjecture in general. A first step in this direction was made in [27], where a very general but non-constructive proof of the existence of maximal loan terms in any even dimension was given. However, the question of whether there is always a positive interest ‘rate’ was left open.

In the second part of this paper, a proof of the quantum interest conjecture for *general* energy distributions in two-dimensional massless scalar field theory is given. Firstly, we explain how a theorem of Simon [28], regarding the existence of negative eigenvalues to

the Hamiltonian, can be used to prove the existence of quantum interest for general energy distributions. We would then use our previous results on square pulses, together with the min-max principle [29], to prove that the interest owed increases with the term of the loan. Finally, we use the same method to give an alternative proof of the existence of a maximum loan term in this case, and furthermore, obtain an upper bound for this loan term.

2. Double square pulses

In this section, we shall investigate the case of square pulses in detail using the eigenvalue approach of [27]. In Sec. 2.1, we obtain a bound on the maximal separation between the negative square pulse and the compensating positive pulse (which need not be square). Then in Sec. 2.2, we determine the minimum quantum interest ‘rate’ required in the case when the positive pulse is also square.

2.1. Pulse separation

Consider a negative-energy square pulse starting at time $t = 0$, of magnitude ρ_1 and duration a . Let us suppose it is followed by an infinite wall of positive energy at time $T \geq a$. This would allow us to find an upper bound on the time separation $T - a$ between the negative energy pulse and the compensating positive one.

The corresponding quantum-mechanical potential $V(t) \equiv 6\pi\rho(t)$ is shown in Fig. 1, where we have set $V_1 \equiv 6\pi\rho_1$. It is divided along the time axis into the four regions as indicated. We would now like to find the conditions for the existence of normalisable bound-state wavefunctions Ψ to the Schrödinger equation

$$H^{(1)}\Psi = -k^2\Psi, \quad (2.1)$$

with energy eigenvalue $-k^2$.

In region I, a solution which vanishes at infinity is

$$\Psi_I(t) = e^{kt}, \quad (2.2)$$

where k is assumed to be positive. Note that this solution is defined up to a normalisation constant which has not been included, as it will not affect the following arguments. Matching (2.2) across the boundaries into regions II and III in the standard way, we obtain the

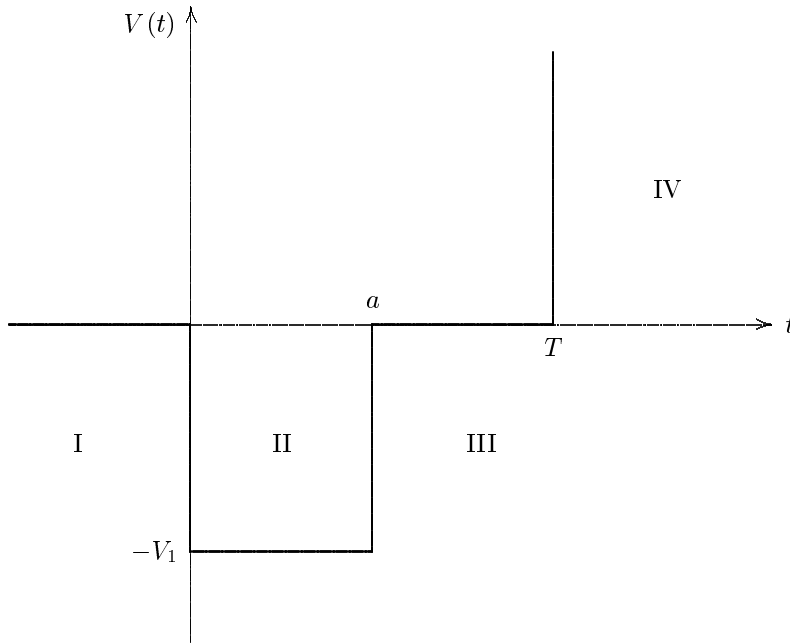


Figure 1: Quantum-mechanical potential corresponding to a negative-energy square pulse followed by an infinite positive-energy wall.

respective solutions:

$$\Psi_{\text{II}}(t) = \cos \omega t + \frac{k}{\omega} \sin \omega t, \quad (2.3)$$

and

$$\Psi_{\text{III}}(t) = \left(\cos \omega a + \frac{k}{\omega} \sin \omega a \right) \cosh k(t - a) + \left(\cos \omega a - \frac{\omega}{k} \sin \omega a \right) \sinh k(t - a), \quad (2.4)$$

where $\omega \equiv \sqrt{V_1 - k^2}$. For bound states to occur, we require $k < \sqrt{V_1}$ (see, e.g., [30]).

The infinite potential at $t = T$ implies that $\Psi_{\text{III}}(t)$ has to vanish at this point. This condition is equivalent to

$$T - a = \frac{1}{k} \tanh^{-1} F(k), \quad (2.5)$$

where

$$F(k) \equiv \frac{k}{\omega} \frac{\cos \omega a + \frac{k}{\omega} \sin \omega a}{\sin \omega a - \frac{k}{\omega} \cos \omega a}. \quad (2.6)$$

Since the left-hand side of (2.5) is non-negative, we must have $0 \leq F(k) < 1$.

The requirement for QI-compatibility is precisely that there is *no* k which satisfies (2.5); we now work out the conditions for this to be so. It can be checked that the graph of $\frac{1}{k} \tanh^{-1} F(k)$ consists of disjoint branches (similar to that of $\tan k$, for example), each of

which is monotonically increasing. The proof of the latter fact, being somewhat technical, is relegated to Appendix A. Thus, a necessary condition for the non-existence of a solution $k \in (0, \sqrt{V_1})$ to (2.5) is

$$\begin{aligned} T - a &\leq \lim_{k \rightarrow 0} \left(\frac{1}{k} \tanh^{-1} F(k) \right) \\ &= \frac{\cot(a\sqrt{V_1})}{\sqrt{V_1}}. \end{aligned} \quad (2.7)$$

Furthermore, we need to ensure that a solution does not come from the *second* branch of $\frac{1}{k} \tanh^{-1} F(k)$. This is equivalent to demanding that at no point in the region $(0, \sqrt{V_1})$ does $\frac{1}{k} \tanh^{-1} F(k)$ vanish, which translates to the requirement that

$$\tan \omega a \neq -\frac{\omega}{k}, \quad (2.8)$$

for $0 < \omega < \sqrt{V_1}$. It is straightforward to solve (2.8) graphically: plotting the graphs of $\tan \omega a$ and $-\frac{\omega}{\sqrt{V_1 - \omega^2}}$ together against ω , it is clear that they do not intersect if and only if

$$a\sqrt{V_1} \leq \frac{\pi}{2}. \quad (2.9)$$

As a consistency check, note that (2.9) ensures that the right-hand side of (2.7) is always non-negative.

Thus, we have found that demanding QI-compatibility of the negative-energy square pulse imposes the two conditions (2.9) and (2.7), which can respectively be written as

$$a\sqrt{\rho_1} \leq \sqrt{\frac{\pi}{24}}, \quad (2.10)$$

$$T - a \leq \frac{\cot(a\sqrt{6\pi\rho_1})}{\sqrt{6\pi\rho_1}}. \quad (2.11)$$

The first condition can be interpreted as a constraint on the magnitude and duration of the negative square pulse: the more negative it is, the shorter its duration must be. Such a result is to be expected from the quantum inequalities. We recall that there is, however, no corresponding result for the case of a negative δ -function pulse [27]. Again, this is not surprising because the quantum inequalities do not place any restrictions on how negative a pulse can be, if it is localised to a definite point in time.

The second condition (2.11) places an upper bound on the time separation $T - a$ between the negative square pulse and the subsequent positive one. Note that $T - a$ depends inversely on ρ_1 and also on a : the more negative the square pulse is, or the longer its duration is, the shorter the time interval must be before the compensating positive pulse arrives. This is entirely consistent with the notion of quantum interest.

Finally, note that if we take the limit $a \rightarrow 0$ such that $B \equiv a\rho_1$ remains finite, the condition (2.11) gives the maximum time separation

$$T_{\max} = \frac{1}{6\pi B}. \quad (2.12)$$

This agrees with the corresponding result obtained in [27] for a negative δ -function pulse of magnitude B .

2.2. Quantum interest ‘rate’

We now turn to the quantum interest ‘rate’, i.e., by how much must a positive square pulse *over*-compensate the negative one. To study this, we need to replace the infinite wall of Sec. 2.1 by a positive square pulse of magnitude ρ_2 , starting at time T and lasting for a duration b . The corresponding potential $V(t)$ is shown in Fig. 2, where as usual, $V_2 \equiv 6\pi\rho_2$. There are now five different regions in time to consider when solving the Schrödinger equation (2.1).

The solutions in regions I, II and III are the same as those derived in Sec. 2.1. Matching the solution $\Psi_{\text{III}}(t)$ across the boundary $t = T$ into region IV gives

$$\Psi_{\text{IV}}(t) = C \cosh \lambda(t - T) + C' \sinh \lambda(t - T), \quad (2.13)$$

where we have defined the constants $\lambda \equiv \sqrt{V_2 + k^2}$, and

$$C \equiv \left(\cos \omega a + \frac{k}{\omega} \sin \omega a \right) \cosh k(T - a) + \left(\cos \omega a - \frac{\omega}{k} \sin \omega a \right) \sinh k(T - a), \quad (2.14a)$$

$$C' \equiv \frac{k}{\lambda} \left[\left(\cos \omega a + \frac{k}{\omega} \sin \omega a \right) \sinh k(T - a) + \left(\cos \omega a - \frac{\omega}{k} \sin \omega a \right) \cosh k(T - a) \right]. \quad (2.14b)$$

Similarly, matching this solution across the boundary $t = T + b$ into region V gives

$$\Psi_{\text{V}}(t) = D \cosh k(t - T - b) + D' \sinh k(t - T - b), \quad (2.15)$$

where

$$D \equiv C \cosh \lambda b + C' \sinh \lambda b, \quad D' \equiv \frac{\lambda}{k} (C \sinh \lambda b + C' \cosh \lambda b). \quad (2.16)$$

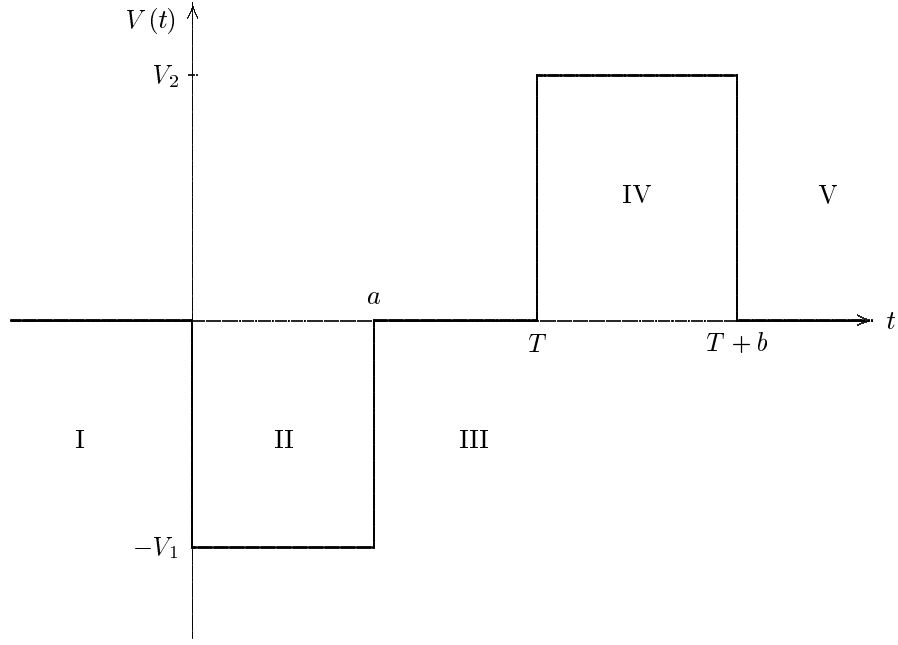


Figure 2: Quantum-mechanical potential corresponding to the case of double square pulses.

Now, to prevent $\Psi_V(t)$ from blowing up as $t \rightarrow \infty$, we require $D + D' = 0$. We obtain, after some calculation,

$$D + D' = (\omega \sin \omega a - k \cos \omega a) g_1(k) G(k), \quad (2.17)$$

where we have defined

$$G(k) \equiv \frac{1}{k} \left(F(k) - \frac{g_2(k)}{g_1(k)} \right), \quad (2.18)$$

and

$$g_1(k) \equiv \left(\frac{\lambda}{k} \cosh k(T-a) + \frac{k}{\lambda} \sinh k(T-a) \right) \sinh \lambda b + e^{k(T-a)} \cosh \lambda b, \quad (2.19a)$$

$$g_2(k) \equiv \left(\frac{\lambda}{k} \sinh k(T-a) + \frac{k}{\lambda} \cosh k(T-a) \right) \sinh \lambda b + e^{k(T-a)} \cosh \lambda b. \quad (2.19b)$$

Note that both $g_1(k)$ and $g_2(k)$ are manifestly positive functions.

For QI-compatibility, we require that normalisable bound-state wavefunctions to (2.1) do *not* exist. In other words, we need to find the conditions under which $D + D' \neq 0$ for all $k \in (0, \sqrt{V_1})$. Now, it can be seen that the right-hand side of (2.17) vanishes if and only if $G(k)$ vanishes. In Appendix B, it is shown that $G(k) > G(0)$ for all $k \in (0, \sqrt{V_1})$. Thus, we

have QI-compatibility if and only if $G(0)$ is non-negative. Since,

$$\lim_{k=0} G(k) = \frac{\cot(a\sqrt{V_1})}{\sqrt{V_1}} - \frac{\coth(b\sqrt{V_2})}{\sqrt{V_2}} - (T - a), \quad (2.20)$$

this means that

$$\begin{aligned} T - a &\leq \frac{\cot(a\sqrt{V_1})}{\sqrt{V_1}} - \frac{\coth(b\sqrt{V_2})}{\sqrt{V_2}} \\ &= \frac{\cot(a\sqrt{6\pi\rho_1})}{\sqrt{6\pi\rho_1}} - \frac{\coth(b\sqrt{6\pi\rho_2})}{\sqrt{6\pi\rho_2}}. \end{aligned} \quad (2.21)$$

The constraint (2.21) is clearly a generalisation of (2.11). However, it now plays two roles: not only does it provide an upper bound for the time separation between the two pulses, it can also tell us how much the positive pulse must over-compensate the negative one if it is rewritten in the form

$$\sqrt{V_2} \tanh(b\sqrt{V_2}) \geq \left(\frac{\cot(a\sqrt{V_1})}{\sqrt{V_1}} - (T - a) \right)^{-1}. \quad (2.22)$$

Let us highlight some implications of this constraint:

1. Since $x < \tan x$ and $\tanh x < x$ for $0 < x < \frac{\pi}{2}$, we may deduce, from (2.21), the following sequence of inequalities:

$$aV_1 < \sqrt{V_1} \tan(a\sqrt{V_1}) \leq \sqrt{V_2} \tanh(b\sqrt{V_2}) < bV_2. \quad (2.23)$$

This implies that

$$a\rho_1 < b\rho_2, \quad (2.24)$$

i.e., the area (in the sense of magnitude multiplied by duration) of the positive pulse is strictly greater than the area of the negative pulse.

2. In the limit when the two areas coincide, it follows from (2.23) that

$$\frac{\cot(a\sqrt{V_1})}{\sqrt{V_1}} \rightarrow \frac{\coth(b\sqrt{V_2})}{\sqrt{V_2}}. \quad (2.25)$$

Then, according to (2.21), $T \rightarrow a$, i.e., there is no pulse separation. Conversely, when the separation is increased, the difference between the areas of the positive and negative pulses will also increase.

3. As $a\sqrt{V_1}$ approaches the maximum allowed value $\frac{\pi}{2}$ [c.f. (2.9)], we note from (2.23) that $V_2 \rightarrow \infty$ and from (2.21) that $T \rightarrow a$. Thus, the quantum interest ‘rate’ becomes infinite in this limit, suggesting that it progressively becomes more and more difficult to reach this maximum amount of negative energy. The maximum value itself is, of course, physically impossible to achieve.
4. If we take the limits $a, b \rightarrow 0$ such that $a\rho_1$ and $b\rho_2$ remain finite, we would expect to recover the corresponding results for a pair of δ -function pulses [27]. If we set $B \equiv a\rho_1$ and $(1 + \epsilon)B \equiv b\rho_2$, the constraint (2.21) implies that

$$\epsilon \geq \frac{6\pi BT}{1 - 6\pi BT}, \quad (2.26)$$

which was indeed the result obtained in [27].

3. Proof of quantum interest in two dimensions

In this section, we shall present a general proof of the quantum interest conjecture for minimally coupled, massless scalar fields in two dimensions. This would be done in two parts: firstly, to prove that the positive pulse must always over-compensate the negative pulse; and secondly, to show that the minimum amount of over-compensation increases with the pulse separation, and that there exists a (finite) maximum separation between the pulses.

The first part of the proof follows almost immediately from a theorem of Simon [28], concerning the existence of negative eigenvalues to the quantum-mechanical Hamiltonian. In a form suitable to our purposes, it reads:

Theorem. Let $V(t)$ obey $\int_{-\infty}^{\infty} (1 + t^2)|V(t)| dt < \infty$, with $V(t)$ not a.e. zero*. Then $H^{(1)} \equiv -d^2/dt^2 + V(t)$ has a negative eigenvalue if

$$\int_{-\infty}^{\infty} V(t) dt \leq 0. \quad (3.1)$$

Since QI-compatibility is equivalent to the condition that there are no negative eigenvalues to $H^{(1)}$, Simon’s theorem implies that if $\int_{-\infty}^{\infty} \rho(t) dt \leq 0$, then $\rho(t)$ is not QI-compatible. In other words, QI-compatibility implies that the positive parts of an energy density must always over-compensate for any negative components.

*Almost everywhere (a.e.) zero: zero everywhere except on a set of measure zero [31].

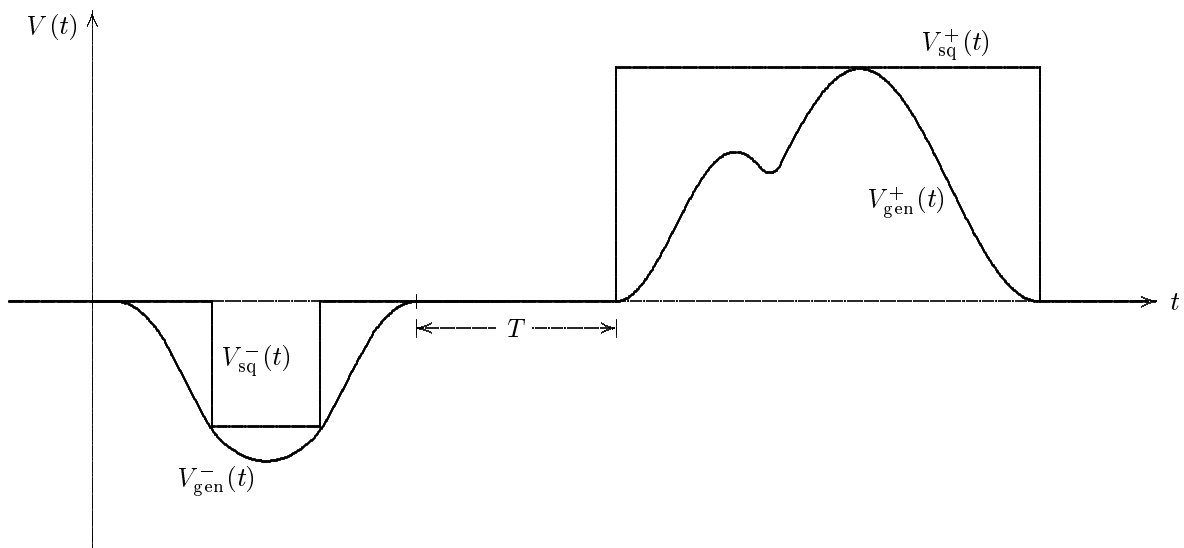


Figure 3: Shape of a general potential $V_{\text{gen}}(t)$, together with the corresponding square potential $V_{\text{sq}}(t)$. The \pm superscripts denote their positive- and negative-energy parts, respectively. T defines the separation between $V_{\text{gen}}^-(t)$ and $V_{\text{gen}}^+(t)$.

Simon's theorem is, in fact, far more general than what we need, since it is valid for potentials (energy-density distributions) that have any number of positive and negative parts, and even for those that are not compactly supported. In order to proceed, however, we shall begin assuming a more specific form of the energy distribution: namely, that it consists of a single continuous negative-energy part $V_{\text{gen}}^-(t)$, followed after time T by a single continuous positive-energy part $V_{\text{gen}}^+(t)$. Furthermore, we demand that both $V_{\text{gen}}^+(t)$ and $V_{\text{gen}}^-(t)$ are compactly supported, i.e., they are each localised to a finite interval in time.

Given such an energy distribution $V_{\text{gen}}(t) = V_{\text{gen}}^+(t) + V_{\text{gen}}^-(t)$, one can always construct a corresponding square-pulse distribution $V_{\text{sq}}(t)$ of the type considered in the previous section, such that

$$V_{\text{sq}}(t) \geq V_{\text{gen}}(t) \quad (3.2)$$

for all t . An example is illustrated in Fig. 3. The positive square pulse $V_{\text{sq}}^+(t)$ is uniquely determined by the requirement that its height is equal to the maximum height of $V_{\text{gen}}^+(t)$, and that its width coincides with the interval on which $V_{\text{gen}}^+(t)$ is non-zero. The choice of the negative square pulse $V_{\text{sq}}^-(t)$ is essentially free, but we shall choose it to be the one with the largest possible area that can 'lie within' $V_{\text{gen}}^-(t)$.

Because of the condition (3.2), the min-max principle [29] can be used to show that the number of bound states of $V_{\text{sq}}(t)$ is always less than or equal to the number of bound states

of $V_{\text{gen}}(t)$. This result is intuitively clear: the potential well of $V_{\text{sq}}(t)$ is shallower, and the potential barrier higher, than that of $V_{\text{gen}}(t)$, making the existence of bound states less likely for the former. QI-compatibility of $V_{\text{gen}}(t)$ therefore implies the QI-compatibility of $V_{\text{sq}}(t)$.

Now, suppose the height and width of $V_{\text{gen}}^+(t)$ do not increase as the separation T between the pulses is increased. This implies that the corresponding square-pulse potential $V_{\text{sq}}(t)$ will also remain QI-compatible as T is increased. However, we know from (2.22) that this cannot be true; $V_{\text{sq}}^+(t)$ has to increase in area as T is increased. Thus we have a contradiction.

A similar proof-by-contradiction can be used to show the existence of a maximum separation T . Suppose T can be made arbitrarily large, yet a QI-compatible $V_{\text{gen}}(t)$ is possible for some sufficiently large positive pulse. This implies that a QI-compatible $V_{\text{sq}}(t)$ can also be found. But we know from (2.7) that the latter is false; a maximum T exists beyond which $V_{\text{sq}}(t)$ is always QI-incompatible regardless of how large the positive pulse is.

This completes the proof of the quantum interest conjecture.

We end with the remark that it is possible to obtain an upper bound on the maximal pulse separation for $V_{\text{gen}}(t)$ using the results of Sec. 2, but, of course, it will not be optimal in general. This bound on T is given by (2.21), where V_1 and a [V_2 and b] are the height and width of $V_{\text{sq}}^-(t)$ [$V_{\text{sq}}^+(t)$], respectively. In view of this, a tighter bound would be obtained if the area of $V_{\text{sq}}^-(t)$ was maximised (as was done above).

4. Conclusion

In this paper, the eigenvalue approach was used to study the problem the quantum interest in two-dimensional minimally coupled, massless scalar field theory. The key to this approach lies in turning the problem into a more familiar and well-studied one, namely that of quantum mechanics on a line. We first explained in detail how the latter viewpoint can be used to investigate the case of double square energy pulses. We then generalised these results, using certain well-known properties of quantum-mechanical Hamiltonians, to obtain a proof of the quantum interest conjecture for general energy-density distributions.

The next obvious course of pursuit would be to extend the results of this paper from two to four dimensions. The main difference is that $H^{(2)}$, given by (1.6), is now a fourth-order differential operator, and so there is no longer a direct connection with the Hamiltonian in quantum mechanics. Still, it should be possible to consider specific energy-density distributions, such as that of double square pulses, by solving the appropriate fourth-order

Schrödinger-type equation. The results should be interesting, especially in the light of the discovery in [27] that the case of double δ -function pulses in four dimensions can never be QI-compatible, no matter how close they are together.

Ultimately, we would want to prove the quantum interest conjecture for massless scalar fields in four dimensions. Since the existence of a maximum loan term has already been established in general [27], it remains to show that the interest ‘rate’ is positive, perhaps using a similar strategy as in this paper. But to do so, a detailed study of the spectrum of fourth-order Schrödinger-type operators, and in particular an appropriate generalisation of Simon’s theorem [28], is first needed.

Appendix A

In this appendix, we show that $\frac{1}{k} \tanh^{-1} F(k)$ is a (piecewise) monotonically increasing function of k . It will be sufficient to consider only the case $F(k) \geq 0$, although this result is generally valid.

Firstly, note that if we define $H(x) \equiv \tanh^{-1} x$ and $H'(x) \equiv dH(x)/dx$, the inequality

$$x \geq \frac{H(x)}{H'(x)} \quad (\text{A.1})$$

is satisfied for $0 \leq x < 1$. Setting $x = F(k)$, this can be used to show that

$$\begin{aligned} \frac{d}{dk} \left(\frac{1}{k} H(F(k)) \right) &\geq H'(F(k)) \left(\frac{F'(k)}{k} - \frac{F(k)}{k^2} \right) \\ &= H'(F(k)) \frac{d}{dk} \left(\frac{1}{k} F(k) \right). \end{aligned} \quad (\text{A.2})$$

Since $H'(F(k)) > 0$, the proof that $\frac{1}{k} \tanh^{-1} F(k)$ is monotonically increasing is reduced to showing that $\frac{1}{k} F(k)$ is monotonically increasing.

Now, the derivative of $\frac{1}{k} F(k)$ is

$$\frac{kV_1\omega a + k(V_1 - 2k^2) \sin \omega a \cos \omega a + \omega^3 + 2\omega k^2 \sin^2 \omega a}{\omega^5 (\sin \omega a - \frac{k}{\omega} \cos \omega a)^2}. \quad (\text{A.3})$$

But the non-negativity of $F(k)$ implies that

$$(V_1 - 2k^2) \sin \omega a \cos \omega a \geq \omega k (\cos^2 \omega a - \sin^2 \omega a), \quad (\text{A.4})$$

and this inequality can be used in (A.3) to show that it is a positive function of k . We conclude that $\frac{1}{k}F(k)$, and therefore $\frac{1}{k}\tanh^{-1}F(k)$, is a monotonically increasing function of k when $0 \leq F(k) < 1$.

Appendix B

Given the function $G(k)$ as defined in (2.18), we would like to show that $G(k) > G(0)$ for all $k \in (0, \sqrt{V_1})$. To this end, we shall instead prove that

$$G(k) \geq \frac{\cot \omega a}{\omega} - \frac{\coth \lambda b}{\lambda} - (T - a), \quad (\text{B.1})$$

which is a slightly stronger result as the right-hand side is an increasing function of $k \in (0, \sqrt{V_1})$, that is equal to $G(0)$ when $k = 0$.

Since $\frac{1}{k}F(k) > \frac{1}{\omega} \cot \omega a$ for $0 < \omega a < \frac{\pi}{2}$, the problem is reduced to showing that

$$\frac{g_2(k)}{g_1(k)} \leq k \left(T - a + \frac{\coth \lambda b}{\lambda} \right). \quad (\text{B.2})$$

This relation can be rewritten as

$$\begin{aligned} & \lambda^2 (k(T - a) \cosh k(T - a) - \sinh k(T - a)) \\ & + k^2 (k(T - a) \sinh k(T - a) - \cosh k(T - a)) \\ & + \lambda k e^{k(T-a)} \coth \lambda b \left(k(T - a) + \frac{k}{\lambda} \coth \lambda b \right) - \lambda k \coth \lambda b \sinh k(T - a) \\ & + \frac{k^3}{\lambda} \coth \lambda b \sinh k(T - a) \geq 0. \end{aligned} \quad (\text{B.3})$$

Since the first and last terms on the left-hand side are non-negative, we only need to check that

$$\begin{aligned} & k^2 (k(T - a) \sinh k(T - a) - \cosh k(T - a)) \\ & + \lambda k e^{k(T-a)} \coth \lambda b \left(k(T - a) + \frac{k}{\lambda} \coth \lambda b \right) - \lambda k \coth \lambda b \sinh k(T - a) \geq 0. \end{aligned} \quad (\text{B.4})$$

Now, note that $x \sinh x - \cosh x \geq -1$, for any $x \geq 0$. This implies that the left-hand side of (B.4) is

$$\begin{aligned} & \geq \lambda k e^{k(T-a)} \coth \lambda b \left(k(T - a) + \frac{k}{\lambda} \coth \lambda b \right) - \lambda k \coth \lambda b \sinh k(T - a) - k^2 \\ & = \lambda k \coth \lambda b (k(T - a) e^{k(T-a)} - \sinh k(T - a)) + k^2 (e^{k(T-a)} \coth^2 \lambda b - 1) \\ & \geq 0, \end{aligned} \quad (\text{B.5})$$

where we have used the relations $\coth x \geq 1$, $xe^x - \sinh x \geq 0$, and $e^x \geq 1$, for any $x \geq 0$, to obtain the final inequality.

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